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On the Dynamics and Ergodic Properties of the *XY* Model

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The return to equilibrium is investigated for one-dimensional (one-sided) chain of the XY model. The initial state is taken to be the Gibbs state for the sum of the Hamiltonian for the XY model of length N and a perturbation by a uniform magnetic field acting on the first n sites. The time evolution under the unperturbed XY model Hamiltonian is studied for the expectation value of the average magnetization of the same first n sites in the infinitely extended system (i.e., after taking the limit $N \to \infty$). It is found that the return to equilibrium occurs for a finite-size perturbation (i.e., for a fixed n), while it does not occur for an infinite-size perturbation (i.e., the limit $n \to \infty$ is taken simultaneously as $N \to \infty$). A certain twisted asymptotic Abelian property of the XY model is shown and used as a technical tool.

KEY WORDS: XY model; ergodic; return to equilibrium; time evolution; asymptotic Abelian.

1. INTRODUCTION

The dynamics of the XY model[1] received much attention recently[2] as an exactly soluble model with a nontrivial many-body interaction. It is known[3] that the magnetization in the z direction shows a nonergodic behavior and contains an explicit "memory" function. However, a singly perturbed state of the system returns to equilibrium,[4] which leads to the interpretation that the XY chain acts like a "heat bath" on the local impurity. A natural question arises in view of these results: How big can

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the "impurity" be with the return to equilibrium still taking place? It is the purpose of this note to shed light on this question.

It is found that the system returns to thermal equilibrium if the perturbation is of a finite size, while this is not the case if the perturbation is of an infinite size (however small its size be in comparison with the total size of the system). This result is shown both by an explicit computation and by an application of a general theory of an infinite system together with an explicit verification of the twisted asymptotic Abelian property of the time evolution (either on the whole algebra of observables or on a subalgebra, depending on the value of a parameter of the model).

2. FORMULATION

We consider the algebra $\mathfrak{A}$ of Pauli spin operators $\sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)}$ $(j = 1, 2, \ldots)$. The XY model Hamiltonian for N sites is

$$H_0^{(N)} = J \sum_{j=1}^{N-1} \left[(1+\gamma)\sigma_x^{(j)}\sigma_x^{(j+1)} + (1-\gamma)\sigma_y^{(j)}\sigma_y^{(j+1)} \right] \tag{2.1}$$

The perturbation we shall consider is the external magnetic field on the first n sites given by

$$H_p^{(n)} = h \sum_{j=1}^{n} \sigma_z^{(j)} \tag{2.2}$$

Here J and h are constants.

We imagine that the perturbation $H_p^{(n)}$ has been acting for $t \leqslant 0$ and we are having the thermal equilibrium state for $H_0^{(N)} + H_p^{(n)}$ at time 0, with the density matrix given by

$$\rho_{N,n} = Z_{N,n}^{-1} \exp\left\{ -\beta \left[H_0^{(N)} + H_p^{(n)} \right] \right\} \tag{2.3}$$

where Z_N is the partition function defined by $\mathrm{tr}_N \rho_{N,n} = 1$. ("$\mathrm{tr}_N$" indicates the trace for Pauli spin matrices at the first N sites.) The expectation value of an operator Q in this state is given by

$$\langle Q \rangle_{N,n} = \mathrm{tr}_N(\rho_{N,n} Q) \tag{2.4}$$

We shall be interested in the behavior of the expectation values under the time evolution without the perturbation, given by

$$\langle Q(t)_N \rangle_{N,n} = \mathrm{tr}\left[\rho_{N,n} Q(t)_N \right] \tag{2.5}$$

$$Q(t)_N = e^{itH_0^{(N)}} Q e^{-itH_0^{(N)}} \tag{2.6}$$

In particular, we focus our attention to the average magnetization per impurity given by

$$\bar{m}_t^{N,n} = n^{-1} \sum_{j=1}^{n} \left\langle \sigma_z^{(j)}(t)_N \right\rangle_{N,n} \tag{2.7}$$

We investigate the behavior of this quantity in the following two cases.

(A) First the thermodynamic limit $N \to \infty$ is taken. Then the infinite time limit $t \to +\infty$ is taken with n finite and fixed.

(B) First the limit $N \to \infty$ with $n = n_N$ being a function of N tending also to infinity. (Apart from $N \geqslant n_N \geqslant 1$ and $\lim_{N\to\infty} n_N = \infty$, no restriction is imposed.) Then the infinite time limit $t \to \infty$ is taken.

For the case (A), we have

$$\lim_{t\to\infty} \lim_{N\to\infty} \langle Q(t) \rangle_{N,n} = \lim_{N\to\infty} \langle Q \rangle_{N,0} \tag{2.8}$$

for arbitrary Q (independent of N), where $\langle Q \rangle_{N,0}$ is the expectation value without perturbation H_p^N and the thermodynamic limit on the right-hand side exists. In particular, for $Q = n^{-1} \sum_{j=1}^{n} \sigma_z^{(j)}$, we have the result that the average magnetization per impurity returns to its (unperturbed) equilibrium value.

For the case (B), we have

$$\lim_{N\to\infty} \bar{m}_t^{N,n_N} = \lim_{N\to\infty} \bar{m}_t^{N,N} \tag{2.9}$$

for all values of γ, where the thermodynamic limit exists and is the same as the behavior of the time evolution of a single z-spin (say at the origin) in two-sided one-dimensional chain where the time evolution is by the two-sided XY-model Hamiltonian (infinitely extended) without the perturbation and the expectation is taken in the equilibrium state for the (infinitely extended two-sided) perturbed Hamiltonian. The right-hand side of (2.9) does not return to the equilibrium value as indicated in the Introduction. (Also shown in Section 7.2.)

3. CONSEQUENCES OF A GENERAL THEORY FOR CASE A WITH $\gamma = 0$

In this section, we shall be using general results on equilibrium states of infinitely extended systems as formulated in the C^*-algebra approach to quantum statistical mechanics of spin lattice systems. (For example, see Ref. 5.)

For any element a in the C^*-algebra $\mathfrak{A}$ of Pauli spins in a one-sided infinite chain, we consider the time evolution given by

$$\alpha_t(a) = \lim_{N\to\infty} e^{itH_0^{(N)}} a e^{-itH_0^{(N)}} \tag{3.1}$$

where the limit is known to exist in the operator norm.[6,7]

Let Θ be the automorphism of $\mathfrak{A}$ representing 180° rotation of spins at all lattice sites around the z direction: $\Theta(\sigma_x^{(j)}) = -\sigma_x^{(j)}$, $\Theta(\sigma_y^{(j)}) = \sigma_y^{(j)}$, and $\Theta(\sigma_z^{(j)}) = \sigma_z^{(j)}$ for all $j = 1, 2, \ldots$. Since $H_0^{(N)}$ is invariant under Θ, the time evolution α_t commutes with Θ.

In the next section, we establish the following twisted asymptotic Abelian property of α_t when $\gamma = 0$. (The "twisting" refers to the presence of Θ.)

Lemma 1. For any $a, b \in \mathfrak{A}$,

$$\lim_{t\to\infty} \|\Theta(a)\alpha_t(b) - \alpha_t(b)a\| = 0 \qquad \text{if} \quad \Theta(b) = -b \tag{3.2}$$

$$\lim_{t\to\infty} \|[a, \alpha_t(b)]\| = 0 \qquad \text{if} \quad \Theta(b) = b \tag{3.3}$$

On the basis of this property, we obtain the following twisted version of a known result:[8]

Lemma 2. For any factor representation π of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$,

$$\operatorname*{w-lim}_{t\to\infty} \{\pi(\alpha_t(a)) - \omega_\Phi(\pi(\alpha_t(a)))\mathbb{1}\} = 0 \qquad \text{if} \quad \Theta(a) = a \tag{3.4}$$

for any unit vector $\Phi \in \mathscr{H}$, where $\mathbb{1}$ is the identity operator, $\omega_\Phi(x) = (\Phi, x\Phi)$ and w-lim denotes the limit in the weak operator topology.

If there is a unit cyclic vector Φ such that $\varphi \equiv \omega_\Phi \circ \pi$ is Θ-invariant [i.e., $\varphi(\Theta(a)) = \varphi(a)$ for any $a \in \mathfrak{A}$], then

$$\operatorname*{w-lim}_{t\to\infty} \{\pi(\alpha_t(a)) - \varphi(\alpha_t(a))\mathbb{1}\} = 0 \tag{3.5}$$

for any $a \in \mathfrak{A}$. In other words,

$$\lim_{t\to\infty} \{(\Psi, \pi(\alpha_t(a))\Psi) - \varphi(\alpha_t(a))\} = 0 \tag{3.6}$$

for any unit vector $\Psi \in \mathscr{H}$ and any $a \in \mathfrak{A}$.

Proof. If $\Theta(a) = a$, (3.3) implies that any weak accumulation point of

$$\pi(\alpha_t(a) - \omega_\Phi(\alpha_t(a))\mathbb{1} \tag{3.7}$$

as $t \to \infty$ commutes with any element of $\pi(\mathfrak{A})$ and hence is in the center of $\pi(\mathfrak{A})''$. By the assumption that π is a factor representation, the center is trivial and hence any weak accumulation point of (3.7) must be $c\mathbb{1}$ with the constant c determined to be 0 by the following computation:

$$c = \omega_\Phi(c\mathbb{1}) = \lim_{t\to\infty} \omega_\Phi(\pi(\alpha_t(a)) - \omega_\Phi(\pi(\alpha_t(a)))\mathbb{1}) = 0$$

By the compactness of the unit ball of $\pi(\mathfrak{A})''$ relative to the weak operator topology, this implies (3.4).

If there is a unit cyclic vector Φ such that $\varphi = \omega_\Phi \circ \pi$ is Θ-invariant, then there is a unique unitary operator V satisfying $V\pi(a)\Phi = \pi(\Theta a)\Phi$ and it satisfies $V\pi(a)V^* = \pi(\Theta a)$. Thus Θ can be extended to $\pi(\mathfrak{A})''$: $\hat{\Theta}(x) = VxV^*$ for $x \in \pi(\mathfrak{A})''$.

Consider $b \in \mathfrak{A}$ satisfying $\Theta(b) = -b$. Take an accumulation point x of $\pi(\alpha_t(b))$ as $t \to \infty$. Since Θ commutes with α_t,

$$VxV^* = \lim_\nu \pi\big(\Theta\{\alpha_{t_\nu}(b)\}\big) = -\lim_\nu \pi\big(\alpha_{t_\nu}(b)\big) = -x \tag{3.8}$$

for some net t_ν. On the other hand, (3.2) implies

$$\hat{\Theta}(y)x = xy \tag{3.9}$$

for $y \in \pi(\mathfrak{A})$ and hence for $y \in \pi(\mathfrak{A})''$. Setting $y = x^*$ and using $\hat{\Theta}(x^*) = \hat{\Theta}(x)^* = -x^*$, we obtain

$$-x^*x = xx^* \tag{3.10}$$

Since $xx^* \geqslant 0$ and $x^*x \geqslant 0$, we obtain $x = 0$. Hence by the same compactness as before,

$$\operatorname*{w-lim}_{t\to\infty} \pi(\alpha_t(a)) = 0 = \varphi(\alpha_t(a)) \tag{3.11}$$

if $\Theta a = -a$, where the last equality follows from

$$\varphi(\alpha_t(a)) = \varphi(\Theta\{\alpha t(a)\}) = \varphi(\alpha_t(\Theta(a))) = -\varphi(\alpha_t(a)) \tag{3.12}$$

Combining with (3.4), we obtain (3.5) because any $a \in \mathfrak{A}$ is a sum $a = a_+ + a_-$ of $a_\pm = [a \pm \Theta(a)]/2$ satisfying $\Theta(a_\pm) = \pm a_\pm$. ∎

The existence of the thermodynamic limits

$$\varphi^{(n)}(Q) = \lim_{N\to\infty} \langle Q \rangle_{N,n} \tag{3.13}$$

is known (Ref. 9 for any finite-range interaction in one dimension including the present case, and Refs. 10 and 11 for more general one-dimensional system), and it is the unique state satisfying the KMS condition at β for the time evolution

$$\alpha_t^{(n)}(a) = \lim_{N\to\infty} \exp\left[i\left(H_0^{(N)} + H_p^{(n)}\right)t\right] a \exp\left[-i\left(H_0^{(N)} + H_p^{(n)}\right)t\right] \tag{3.14}$$

Since (3.14) is an inner perturbation of $\alpha_t = \alpha_t^{(0)}$ by a relative Hamiltonian $H_p^{(n)} \in \mathfrak{A}$, the unique KMS state $\varphi^{(n)}$ is given by a vector $\Phi^{(n)}$ [usually denoted as $\Phi(H_p^{(n)})$] in the cyclic representation space $\mathscr{H}_\varphi$ (the GNS representation space) in which a cyclic vector $\Phi = \Phi^{(0)} \in \mathscr{H}_\varphi$ gives the state $\varphi = \varphi^{(0)}$.[12] As the unique KMS state for α_t, which commutes with Θ, φ is Θ-invariant (because $\varphi \circ \Theta$ is again a KMS state which must coincide with the unique KMS state φ) as well as a α_t-invariant and yields a factor representation (as it is an extremal KMS state). Hence (3.6) implies

$$\lim_{t\to\infty}\lim_{N\to\infty}\langle Q(t)_N\rangle_{N,n}=\lim_{t\to\infty}\lim_{N\to\infty}\langle\alpha_t(Q)\rangle_{N,n}=\lim_{t\to\infty}\varphi^{(n)}(\alpha_t(Q))$$
$$=\lim_{t\to\infty}\left(\Phi^{(n)},\pi(\alpha_t(Q))\Phi^{(n)}\right)$$
$$=\lim_{t\to\infty}\varphi(\alpha_t(Q))=\varphi(Q)$$

Namely (2.8) is shown.

4. PROOF OF TWISTED ASYMPTOTIC ABELIAN PROPERTY FOR THE CASE $\gamma = 0$

Let

$$c_j=\left(\prod_{k=1}^{j-1}\sigma_z^{(k)}\right)\left(\sigma_x^{(j)}-i\sigma_y^{(j)}\right)/2 \tag{4.1}$$

It satisfies the canonical anticommutation relations:

$$\{c_j,c_k\}=0,\qquad \{c_j,c_k^*\}=\delta_{jk} \tag{4.2}$$

where $\{A,B\}=AB+BA$. The original operators are given by

$$\sigma_z^{(j)}=2c_j^*c_j-1 \tag{4.3}$$

$$\sigma_x^{(j)}=\left[\prod_{k=1}^{j-1}(2c_k^*c_k-1)\right](c_j+c_j^*) \tag{4.4}$$

$$\sigma_y^{(j)}=\left[\prod_{k=1}^{j-1}(2c_k^*c_k-1)\right]i(c_j-c_j^*) \tag{4.5}$$

The algebra $\mathfrak{A}$ can be identified as the CAR algebra generated by c_j and c_j^*, $j=1,2,\ldots$.

The Hamiltonian $H_0^{(N)}$ is expressed as

$$H_0^{(N)}=(-2J)\sum_{j=1}^{N-1}\left[c_j^*c_{j+1}+c_{j+1}^*c_j+\gamma\left(c_j^*c_{j+1}^*+c_{j+1}c_j\right)\right] \tag{4.6}$$

If we write $c(f)=\sum_{n=1}^{\infty}f_nc_n$ and $c^*(f)=\sum_{n=1}^{\infty}f_nc_n^*$ for $f=(f_1,f_2,\ldots)\in l_2$, both series converge in norm and

$$\left[H_0^{(N)},c^*(f)\right]=(-2J)\left[c^*((U_N+U_N^*)f)-\gamma c((U_N-U_N^*)f)\right] \tag{4.7}$$

$$\left[H_0^{(N)},c(f)\right]=(-2J)\left[\gamma c^*((U_N-U_N^*)f)-c((U_N+U_N^*)f)\right] \tag{4.8}$$

where

$$U_N(f_1,f_2,\ldots,f_N,f_{N+1},\ldots)=(f_2,\ldots,f_N,0,f_{N+1},\ldots) \tag{4.9}$$

$$U_N^*(f_1,f_2,\ldots,f_N,f_{N+1},\ldots)=(0,f_1,\ldots,f_{N-1},f_{N+1},\ldots) \tag{4.10}$$

We now specialize to the case of $\gamma = 0$. (The general case later.) We have

$$\exp\left[iH_0^{(N)}t\right]c^*(f)\exp\left[-iH_0^{(N)}t\right] = c^*\left(\exp\left[-2Ji(U_N + U_N^*)t\right]f\right) \quad (4.11)$$

and hence

$$\alpha_t(c^*(f)) = c^*\left(\exp\left[-2Ji(U + U^*)t\right]f\right) \quad (4.12)$$

where

$$U(f_1, f_2, \ldots) = (f_2, f_3, \ldots) \quad (4.13)$$

$$U^*(f_1, f_2, \ldots) = (0, f_1, \ldots) \quad (4.14)$$

Similarly (or taking the adjoint),

$$\alpha_t(c(f)) = c\left(e^{2Ji(U+U^*)t}f\right) \quad (4.15)$$

Lemma 3. The spectral measure of $U + U^*$ is absolutely continuous.

Proof. For any $f = (f_1, f_2, \ldots)$ with $f_j = 0$ except for a finite number of j's, define the sine transform by

$$\tilde{f}(s) = \sum_{n=1}^{\infty} f_n \sin ns \in L_2([0,\pi], (2/\pi)\, ds) \equiv \mathscr{L} \quad (4.16)$$

Then the closure $\mathscr{F}$ of the map $f \to \tilde{f}$ is known to be a unitary map of l_2 onto $\mathscr{L}$. Furthermore

$$\begin{aligned}\left[(U + U^*)\tilde{f}(s)\right] &= \sum_{n=1}^{\infty} f_n(\sin(n+1)s + \sin(n-1)s)\\ &= 2(\cos s)\tilde{f}(s) \end{aligned} \quad (4.17)$$

Therefore, $U + U^*$ is a multiplication operator of $2\cos s$ and has an absolutely continuous spectrum.

Corollary. For any f_1 and f_2 in l_2 and for any real λ,

$$\lim_{t\to\infty} \left(f_1, e^{i\lambda(U+U^*)t}f_2\right) = 0 \quad (4.18)$$

Proof. By the spectral resolution $(U + U^*) = \int x\, dE(x)$,

$$\left(f_1, e^{i\lambda(U+U^*)t}f_2\right) = \int e^{i\lambda xt}\, d\mu(x) \quad (4.19)$$

where

$$d\mu(x) \equiv (f_1, dE(x)f_2) = \rho(x)\, dx \quad (4.20)$$

for some $\rho \in L_1$ by Lemma 3. By the Riemann–Lebesgue lemma, (4.18) follows. ∎

Proof of Lemma 1. This Corollary, (4.12), (4.15), and the canonical anticommutation relations imply (3.2) when a and b are creation and annihilation operators $c^*(f)$ and $c(g)$. In view of the property $\Theta(c^*(f)) = -c^*(f)$ and $\Theta(c(g)) = -c(g)$, (3.2) and (3.3) for a general a and b follow. ■

5. PROOF OF (2.9)

The proof actually works for a general class of finite-range interactions and is based on a result in Ref. 9 which we first summarize.

Let $H_0^{[k,l]}$ be given by (2.1) with sum over $j = k, k+1, \ldots, l-1$, $H_p^{[k,l]}$ be given by (2.2) with sum over $j = k, k+1, \ldots, l$ and $H^{[k,l]} = H_0^{[k,l]} + H_p^{[k,l]}$, where k and l are integrers satisfying $k < l$ and we now consider the Gibbs state for a two-sided chain:

$$\varphi^{k,l}(x) = \hat{Z}_{k,l}^{-1} \mathrm{tr}_{[k,l]}\{\exp(-\beta H^{[k,l]})x\} \tag{5.1}$$

with $\hat{Z}_{k,l}$ defined by $\varphi^{k,l}(1) = 1$. This limits

$$\varphi^{k,\infty}(x) = \lim_{l\to\infty} \varphi^{k,l}(x) \tag{5.2}$$

$$\varphi^{-\infty,\infty}(x) = \lim_{k\to\infty} \lim_{l\to\infty} \varphi^{k,l}(x) \tag{5.3}$$

exist and the limits are uniform in the following sense.

Lemma 4. Let $\mathfrak{A}([k,l])$ be the algebra generated by $\boldsymbol{\sigma}^{(j)}$, $j \in [k,l]$ and $x \in \mathfrak{A}([k,l])$. There exist constants c_1 and c_2 independent of k, l, r, and x such that

$$|\varphi^{k,l+r}(x) - \varphi^{k,\infty}(x)| \leqslant \|x\| c_1 e^{-c_2 r} \tag{5.4}$$

$$|\varphi^{k-r,l+r}(x) - \varphi^{-\infty,\infty}(x)| \leqslant \|x\| c_1 e^{-c_2 r} \tag{5.5}$$

States $\varphi^{k,\infty}$ and $\varphi^{-\infty,\infty}$ have the following exponential clustering property.

Lemma 5. $\mathfrak{A}([k,l]^c)$ be the C^*-algebra generated by $\boldsymbol{\sigma}^{(j)}$, $j \notin [k,l]$. Let $r > 0$, $N > l + r$, $-M < k - r$. There exist constants d_1 and d_2 independent of $k, l, r, x \in \mathfrak{A}([k,l]), y_1 \in \mathfrak{A}([l+r, N])$ and $y_2 \in \mathfrak{A}([-M, k-r] \cup [l+r, N])$ satisfying

$$|\varphi^{k,N}(xy_1) - \varphi^{k,N}(x)\varphi^{k,N}(y_1)| \leqslant d_1 \|x\| \, \|y_1\| e^{-d_2 r} \tag{5.6}$$

$$|\varphi^{-M,N}(xy_2) - \varphi^{-M,N}(x)\varphi^{-\infty,\infty}(y_2)| \leqslant d_1 \|x\| \, \|y_2\| e^{-d_2 r} \tag{5.7}$$

Due to the split

$$H_0^{[k,l]} = H_0^{[k,j-1]} + H_0^{[j,l]} + W_j \tag{5.8}$$

$$H_p^{[k,l]} = H_p^{[k,j-1]} + H_p^{[j,l]} \tag{5.9}$$

with $W_j = J\{(1+\gamma)\sigma_x^{(j-1)}\sigma_x^{(j)} + (1-\gamma)\sigma_y^{(j-1)}\sigma_y^{(j)}\}$, and the uniform limit

$$\alpha_t^{[1,\infty]}(x) = \lim_{n\to\infty} \exp\left[i\left(H_0^{N(n)} + H_p^{(n)}\right)t\right] x \exp\left[-i\left(H_0^{N(n)} + H_p^{(n)}\right)t\right] \tag{5.10}$$

[uniformly over the choice of $N(n) \geqslant n$], correlation across a boundary can be expressed in terms of an (explicitly defined) operator in $\mathfrak{A}$ as follows:

Lemma 6. There is an operator $\Lambda_{k,N,n}$ such that

$$\langle Q\rangle_{N,n} = (\varphi^{1,k} \otimes \varphi^{k,N,n})(\Lambda_{k,N,n}^* Q \Lambda_{k,N,n}) \tag{5.11}$$

for all $Q \in \mathfrak{A}([1,k])$, where $k \leqslant n \leqslant N$ and

$$\varphi^{k,N,n}(x) = Z_{k,N,n}^{-1} \operatorname{tr}_{[k+1,N]}\left\{\exp\left[-\beta\left(H_0^{[k+1,N]} + H_p^{[k+1,n]}\right)\right]x\right\} \tag{5.12}$$

with $\varphi^{k,N,n}(1) = 1$,

$$c = \sup_{k,N,n} \|\Lambda_{k,N,n}\| < \infty \tag{5.13}$$

and

$$d\left(\Lambda_{k,N,n}, \mathfrak{A}([k-r, k+r])\right) \leqslant b_1 e^{-b_2 r} \tag{5.14}$$

for some constants b_1 and $b_2 > 0$ independent of k, N, n where

$$d(\Lambda, \mathscr{B}) = \inf\{\|\Lambda - \Lambda'\|; \Lambda' \in \mathscr{B}\} \tag{5.15}$$

Let δ be the one-step shift of the lattice to right given by $\delta(\boldsymbol{\sigma}^{(j)}) = \boldsymbol{\sigma}^{(j+1)}$. It is an isomorphism of $\mathfrak{A}([k,l])$ onto $\mathfrak{A}([k+1, l+1])$ and an automorphism of the C^*-algebra $\hat{\mathfrak{A}}$ generated by all $\boldsymbol{\sigma}^{(j)}$, $-\infty < j < \infty$. Then $\delta^{-n}\alpha_t\delta^n$ is the time translation of the XY model on the semiinfinite interval $[1-n, \infty)$ and $\hat{\alpha}_t = \lim_{n\to\infty} \delta^{-n}\alpha_t\delta^n$ is the time translation for the XY model on the two-sided infinite chain $\mathbb{Z}$.

On the basis of the preceding three lemmas, we can prove the following:

Lemma 7. If $N(n) \geqslant n$ and $N(n) \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \overline{m}_t^{N(n),n} = \varphi^{-\infty,\infty}\left(\hat{\alpha}_t\left(\sigma_z^{(0)}\right)\right) \tag{5.16}$$

Remark. We may also write

$$\lim_{N\to\infty} \overline{m}_t^{N,n(N)} = \lim_{n\to\infty} \overline{m}_t^{N(n),n} \tag{5.17}$$

provided that $n(N)\to\infty$. Since we may take $N(n) = n$ or $n(N) = N$ and the limit is independent of $N(n)$ or $n(N)$, we obtain (2.9).

Proof. Given $\epsilon > 0$, let r_1 be such that

$$b_1 e^{-b_2 r_1} < \epsilon/(12c) \tag{5.18}$$

By Lemma 6, there exists $\Lambda'_{k,N,n} \in \mathfrak{A}([k - r_1, k + r_1])$ satisfying $\|\Lambda'_{k,N,n}\| \leqslant c$ and $\|\Lambda_{k,N,n} - \Lambda'_{k,N,n}\| \leqslant b_1 e^{-b_2 r_1}$. {Take $p(\Lambda'_{k,N,n})$ for a conditional expectation p from $\mathfrak{A}$ onto $\mathfrak{A}([k - r_1, k + r_1])$.}

Let $Q \in \mathfrak{A}([i, j])$, $1 \leqslant i \leqslant j$ and $\|Q\| = 1$. Then

$$|\langle Q\rangle_{N,n} - \varphi^{1,k}(Q\Lambda''_{k,N,n})| < \epsilon/6 \tag{5.19}$$

as long as $j \leqslant k - r_1 - 1$ and $k + r_1 \leqslant n \leqslant N$, where

$$\Lambda''_{k,N,n} = \varphi^{k,N,n}((\Lambda'_{k,N,n})^* \Lambda'_{k,N,n}) \in \mathfrak{A}([k - r_1, k]) \tag{5.20}$$

satisfies $\|\Lambda''_{k,N,n}\| \leqslant c^2$ and [by substituting $Q = 1$ into (5.19)]

$$|1 - \varphi^{1,k}(\Lambda''_{k,N,n})| < \epsilon/6 \tag{5.21}$$

Let r_2 be such that

$$d_1 e^{-d_2 r_2} < \epsilon/(6c^2) \tag{5.22}$$

By Lemma 5, if $j < k - r_1 - r_2$

$$|\varphi^{1,k}(Q\Lambda''_{k,N,n}) - \varphi^{1,k}(Q)\varphi^{1,k}(\Lambda''_{k,N,n})| < \epsilon/6 \tag{5.23}$$

Hence (5.21) and (5.19) imply

$$|\langle Q\rangle_{N,n} - \varphi^{1,k}(Q)| < \epsilon/2 \tag{5.24}$$

as long as $j < k - r_1 - r_2$ and $k + r_1 \leqslant n \leqslant N$.

By the proof of the convergence of the time translation like (5.1) and (5.10), there is an l and $Q_j \in \mathfrak{A}([j - l, j + l)$ for given t such that $\|Q_j\| \leqslant 1$ and

$$\|\alpha_t(\sigma_z^{(j)}) - Q_j\| < \epsilon/16 \tag{5.25}$$

If $1 \leqslant j - l, j + l \leqslant k - r_1 - r_2 - 1$ and $k + r_1 \leqslant n \leqslant N$, then

$$\left|\left\langle \alpha_t(\sigma_z^{(j)})\right\rangle_{N,n} - \varphi^{1,k}\left(\alpha_t(\sigma_z^{(j)})\right)\right| < 5\epsilon/8 \tag{5.26}$$

By the lattice translation, we have

$$\varphi^{1,k}\left(\alpha_t(\sigma_z^{(j)})\right) = \varphi^{1-j,k-j}\left(\delta^{-j}\alpha_t\delta^j(\sigma_z^{(0)})\right) \tag{5.27}$$

Let r_3 be such that

$$c_1 e^{-c_2 r_3} < \epsilon/16 \tag{5.28}$$

By Lemma 4 and (5.25),

$$\left|\varphi^{1-j,k-j}\left(\delta^{-j}\alpha_t\delta^j\left(\sigma_z^{(0)}\right)\right) - \varphi^{-\infty,\infty}\left(\delta^{-j}\alpha_t\delta^j\left(\sigma_z^{(0)}\right)\right)\right| < 3\epsilon/16 \tag{5.29}$$

if $1-j \leqslant -l-r_3$ and $k-j \geqslant l+r_3$. Let j_0 be such that $\|\delta^{-j}\alpha_t\delta^j(\sigma_z^{(0)}) - \hat{\alpha}_t(\sigma_z^{(0)})\| < \epsilon/16$ for $j \geqslant j_0$. Then (5.26), (5.27), and (5.29) imply

$$\left|\left\langle \alpha_t\left(\sigma_z^{(j)}\right)\right\rangle_{N,n} - \varphi^{-\infty,\infty}\left(\hat{\alpha}_t\left(\sigma_z^{(0)}\right)\right)\right| < 7\epsilon/8 \tag{5.30}$$

if $\max(j_0, 1+l+r_3) \leqslant j \leqslant n - \max(l+r_3, r_1+r_2+1) - r_1$ and $n \leqslant N$.

Let n_0 be such that

$$\left[\max(j_0, l+r_3) + r_1 + \max(l+r_3, r_1+r_2+1)\right]/n_0 \leqslant \epsilon/16 \tag{5.31}$$

Then for $N \geqslant n \geqslant n_0$, we obtain

$$\left|\overline{m}_t^{N,n} - \varphi^{-\infty,\infty}\left(\hat{\alpha}_t\left(\sigma_z^{(0)}\right)\right)\right| < \epsilon \tag{5.32}$$

Hence (5.16) holds. ■

6. CASE OF A GENERAL γ WITH LOCALIZED PERTURBATION ($n < \infty$)

We shall use the description of the canonical anticommutation relations in terms of a self-dual CAR algebra.[13] For $f \oplus g^* \in l_2 \oplus l_2^* \equiv L$,

$$B(f \oplus g^*) \equiv c^*(f) + (c^*(g))^* \tag{6.1}$$

where l_2^* is the dual Hilbert space of l_2 and g^* is the functional $g^*(f) = (g, f)$. If we write $\Gamma(f \oplus g^*) = g \oplus f^*$, then $B(h)^* = B(\Gamma h)$. We also have

$$\{B(h_1)^*, B(h_2)\} = (h_1, h_2)$$

where $(f_1 \oplus g_1^*, f_2 \oplus g_2^*) = (f_1, f_2) + (g_2, g_1)$. We shall identify l_2^* with $l_2[(g_1, g_2, \ldots)^*$ with $(\bar{g}_1, \bar{g}_2, \ldots)]$ and $L = l_2 \oplus l_2$ with $l_2 \otimes \mathbb{C}^2$. [Then $c^*(g)^* = c(\bar{g})$.] The Hamiltonian $H_0^{(N)}$ satisfies (4.7) and (4.8) and hence

$$\lim_{N\to\infty}\left[H_0^{(N)}, B(h)\right] = B(-2JK_\gamma h) \tag{6.2}$$

$$K_\gamma = \begin{pmatrix} U + U^*, & \gamma(U - U^*) \\ -\gamma(U - U^*), & -(U + U^*) \end{pmatrix} \tag{6.3}$$

$$\begin{aligned} U(f_1, f_2, \ldots) &= (f_2, f_3, \ldots) \\ U^*(f_1, f_2, f_3, \ldots) &= (0, f_1, f_2, \ldots) \end{aligned} \tag{6.4}$$

We note that $K_\gamma^* = K_\gamma$ and $\Gamma K_\gamma = -K_\gamma\Gamma$. Since K_γ is bounded, it immediately follows that

$$\alpha_t(B(h)) = \lim_{N\to\infty} e^{itH_\delta^{(N)}} B(h) e^{-itH_\delta^{(N)}} = B(e^{-2iJK_\gamma t}h) \tag{6.5}$$

The asymptotic property of α_t for large t is determined by the spectral property of K_γ given by the following:

Lemma 8. The dimension of $\ker K_\gamma$ is 1 and K_γ has an absolutely continuous spectrum on $(\ker K_\gamma)^\perp$, where $\ker K_\gamma$ denotes the kernel of K_γ (i.e., the eigenspace belonging to an eigenvalue 0).

The space $\ker K_\gamma$ is spanned by

$$(f_{\gamma,0} \oplus f_{\gamma,0}) \quad \text{if} \quad 0<\gamma<1 \quad \text{and by} \quad (f_{\gamma,0} \oplus -f_{\gamma,0})$$

if $-1<\gamma<0$ where

$$(f_{\gamma,0})_{2n} = 0 \tag{6.6}$$

$$= (f_{\gamma,e})_{2n-1} = \left(\frac{1-|\gamma|}{1+|\gamma|}\right)^{n-1} \tag{6.7}$$

Proof. Let

$$V = 2^{-1/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then V is unitary and

$$VK_\gamma V^* = \begin{pmatrix} 0 & A_\gamma \\ A_\gamma^* & 0 \end{pmatrix} \tag{6.8}$$

$$A_\gamma = U + U^* - \gamma(U - U^*) \tag{6.9}$$

Thus

$$VK_\gamma^2 V^* = \begin{pmatrix} A_\gamma A_\gamma^* & 0 \\ 0 & A_\gamma^* A_\gamma \end{pmatrix}$$

with

$$A_\gamma^* A_\gamma = (1-\gamma^2)\left[U^2 + (U^*)^2\right] + (1+\gamma^2)(UU^* + U^*U) + 2\gamma[U, U^*] \tag{6.10}$$

$$= 2(1+\gamma^2) + (1-\gamma^2)\left[U^2 + (U^*)^2 - \alpha P\right] \tag{6.11}$$

$$\alpha = (1-\gamma)/(1+\gamma) \tag{6.12}$$

where P is the one-dimensional projection on the vector $(1,0,0\ldots)$ and we have used the identities $UU^* = 1$, $U^*U = 1 - P$.

Let $l_2 \equiv l_2(\mathbb{N}) = l_{2,0} \oplus l_{2,e}$ where $l_{2,0} = l_2(2\mathbb{N}-1)$ and $l_{2,e} = l_2(2\mathbb{N})$ are identified with l_2 such that $f = f_0 \oplus f_e$ with $(f_0)_n = f_{2n-1}$, $(f_e)_n = f_{2n}$. The two subspaces $l_{2,0}$ and $l_{2,e}$ are invariant under $A_\gamma^* A_\gamma$, the restrictions of $A_\gamma^* A_\gamma$ to $l_{2,0}$ and $l_{2,e}$ are equivalent to $2(1+\gamma^2) + (1-\gamma^2)(U + U^* - \alpha P)$ and $2(1+\gamma^2) + (1-\gamma^2)(U+U^*)$. In Section 4, we have already seen that $U + U^*$ has an absolutely continuous spectrum (Lemma 3).

To study the spectrum of $U + U^* - \alpha P$, consider the case $0 < \gamma < 1$, i.e., $0 < \alpha < 1$. Set

$$\begin{aligned} L &= (1+\alpha U^*)(1+\alpha U) = 1 + \alpha(U+U^*) + \alpha^2 U^* U \\ &= 1 + \alpha^2 + \alpha(U + U^* - \alpha P) \end{aligned} \tag{6.13}$$

$$W = L^{-1/2}(1+\alpha U^*) \tag{6.14}$$

Note that $(1+\alpha U^*)$ and hence L have bounded inverses due to $\|\alpha U^*\| = \alpha < 1$. Hence W is unitary. Since L commutes with $U + U^* - \alpha P$,

$$\begin{aligned} (U + U^* - \alpha P)W &= L^{-1/2}(U + U^* - \alpha P)(1 + \alpha U^*) \\ &= L^{-1/2}(1+\alpha U^*)(U+U^*) = W(U+U^*) \end{aligned} \tag{6.15}$$

Therefore $U + U^* - \alpha P$ has the same spectrum as $U + U^*$. Thus, $A_\gamma^* A_\gamma$ has an absolutely continuous spectrum if $0 < \gamma < 1$. (In particular, $\ker A_\gamma = 0$.)

Let $A_\gamma = v|A_\gamma|$ be the polar decomposition of A_γ. Since $\ker A_\gamma = 0$, v is isometric and $A_\gamma A_\gamma^* = v A_\gamma^* A_\gamma v^*$. Therefore the spectrum of $A_\gamma A_\gamma^*$ on $(\ker v^*)^\perp$ is the same as the spectrum of $A_\gamma^* A_\gamma$, i.e., an absolutely continuous spectrum. In addition $A_\gamma A_\gamma^*$ has an eigenvalue 0 on $\ker v^* = \ker A^*$. The eigenvalue equation $A_\gamma^* f = 0$, which is equivalent to $(U + \alpha U^*)f = 0$, can be solved directly and yields the eigenvectors $f_{\gamma,0}$. Thus the lemma is proved for the case $0 < \gamma < 1$.

Since $A_{-\gamma} = A_\gamma^*$, the case $0 > \gamma > -1$ is exactly the same as above if $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ is used instead of $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ in V. ■

If we substitute the above Lemma in place of Lemma 3 in the proof of Lemma 1 discussed in Section 4, we obtain the following.

Lemma 9. Assume $-1 < \gamma < 1$, $\gamma \neq 0$.

(1) The fixed point algebra

$$\mathscr{A}^\alpha = \{a \in \mathscr{A} : \alpha_t(a) = a \text{ for all } t\} \tag{6.16}$$

is a two-dimensional Abelian algebra generated by $s = c^*(f_{\gamma,0}) + c\, c(f_{\gamma,0})$ $(\epsilon = \operatorname{sign}\gamma)$, which satisfies $\Theta(s) = -s$ and $s^2 = \epsilon\|f_{\gamma,0}\|^2 \mathbb{1}$.

(2) The twisted commutant

$$(\mathscr{A}^{\alpha})^{tc} = \{a \in \mathscr{A} : \Theta(a)s = sa\} \tag{6.17}$$

together with $\mathscr{A}^{\alpha}$, algebraically generate $\mathscr{A}$.

(3) For $a \in \mathscr{A}$ and $b \in (\mathscr{A}^{\alpha})^{tc}$, (3.2) and (3.3) hold.

Proof. Let $\mathscr{B}_1 \equiv \mathbb{C}1 + \mathbb{C}s$ and $\mathscr{B}_2$ be the C^*-subalgebra of $\mathscr{A}$ generated by $B(h)$, $h \in (\operatorname{Ker} K_\gamma)^{\perp}$. Any element Q in the subalgebra B of $\mathscr{A}$ algebraically generated by $\mathscr{B}_1$ and $\mathscr{B}_2$ can be written as $Q = Q_0 + sQ_1$ with Q_0, $Q_1 \in \mathscr{B}_2$. Then $2Q_1 = c\|f_{\gamma,0}\|^{-2}[sQ - \Theta(Q)s]$. Hence any norm limit of such Q must be of the same form. Namely $\mathscr{B}_1$ and $\mathscr{B}_2$ algebraically generate $\mathscr{A}$. The same formula shows for $Q \in (\mathscr{B}_1)^{tc}$ that $Q_1 = 0$. Hence $\mathscr{B}_2 = \mathscr{B}_1^{tc}$. Again the same formula and the commutativity of Θ with α_t show that $\mathscr{A}^{\alpha}$ is algebraically generated by $\mathscr{B}_1$ and $\mathscr{B}_2^{\alpha} = \mathscr{B}_2 \cap \mathscr{A}^{\alpha}$. Lemma 8 proves (3.2) and (3.3) for $a \in \mathscr{A}$ and $b \in \mathscr{B}_2$. Since the trace state on $\mathscr{A}$ is faithful, Θ-invariant (invariant under any automorphism), factor state, (6.20) in the next Lemma 9, which is a consequence of (3.3), implies that $\mathscr{B}_2^{\alpha} = C1$ (due to $\alpha_t(a) = a$ for $a \in \mathscr{B}_2^{\alpha}$). Therefore $\mathscr{A}^{\alpha} = \mathscr{B}_1$. ■

Lemma 10. For any factor representation π of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$,

$$\operatorname*{w-lim}_{t\to\infty} \{\pi(\alpha_t(a)) - \omega_\Phi(\pi(\alpha_t(a))1)\} = 0 \tag{6.18}$$

for any $a \in \mathfrak{A}_\gamma^{tc}$ satisfying $\Theta a = a$ and for any unit vector $\Phi \in \mathscr{H}$.

If there is a unit cyclic vector Φ such that $\varphi = \omega_\Phi t$, π is Θ-invariant,

$$\operatorname*{w-lim}_{t\to\infty} \{\pi(\alpha_t(a)) - \varphi(\alpha_t(a))1\} = 0 \tag{6.19}$$

for any $a \in \mathfrak{A}_\gamma^{tc}$. In other words

$$\lim_{t\to\infty} \{(\Psi, \alpha_t(a)\Psi) - \varphi(\alpha_t(a))\} = 0 \tag{6.20}$$

for any unit vector $\Psi \in \mathscr{H}$ and any $a \in \mathfrak{A}_\gamma^{tc}$.

We are now ready to prove (2.8) for $\gamma \notin 0$.

Proof of (2.8). Let $Q = Q_0 + sQ_1$ with Q_0, $Q_1 \in (\mathscr{A}^{\alpha})^{tc}$. As before, $\phi(x) = \lim_{N\to\infty}\langle x\rangle_{N,0}$ is Θ-invariant and gives rise to a factor representation π with a cyclic vector Φ satisfying $\omega_\Phi(\pi(x)) = \phi(x)$. The state $\phi^{(n)}(x) = \lim_{N\to\infty}\langle x\rangle_{N,n}$ is also Θ-invariant and is given by a vector Φ_n in the same space. Thus Lemma 10 implies

$$\lim_{t\to\infty} \phi^{(n)}(\alpha_t(Q)) = \phi(Q_0) + \phi(Q_1)\phi^{(n)}(s) \tag{6.21}$$

By Θ-invariance of $\phi^{(n)}$ and $\Theta(s) = -s$, we obtain $\phi^{(n)}(s) = 0$. Thus

$$\lim_{t\to\infty}\lim_{N\to\infty}\langle Q(t)_N\rangle_{N,n} = \phi(Q_0) \tag{6.22}$$

Let $Q_1 = Q_{1+} + Q_{1-}$, where $Q_{1\pm} = [Q_1 \pm \Theta(Q_1)]/2$ belongs to $(\mathscr{A}^{\alpha})^{tc}$ and satisfies $\Theta(Q_{1\pm}) = \pm Q_{1\pm}$. By Θ-invariance of ϕ, we obtain $\phi(sQ_{1+}) = Q$.

Since $\alpha_t(s) = s$, the KMS condition for ϕ implies $\phi(sQ_{1-}) = \phi(Q_{1-}s) = \phi((sQ_{1-} + Q_{1-}s)/2)$, which vanishes due to (6.18) and $\Theta(Q_{1-}) = -Q_{1-}$. Hence $\phi(Q) = \phi(Q_0)$. Therefore (6.23) implies 2.8). ■

7. THE EQUILIBRIUM STATE φ FOR THE XY MODEL

7.1. Unperturbed Case for One-Sided Infinite Chain

We use the same notation as the preceding section. The state $\varphi(Q) \equiv \lim_{N\to\infty}\langle Q\rangle_{N,0}$ is the unique (α_t, β)-KMS state for $\alpha_t(B(h)) = B(e^{-2JitK_\gamma}h)$. The quasifree state with the following two-point function is easily seen to satisfy the (α_t, β)-KMS condition and hence is φ:

$$\varphi(B(\Gamma h_1)B(h_2)) = \left(h_1, (1 + e^{2J\beta K_\gamma})^{-1}h_2\right) \tag{7.1}$$

(The commutation relation and positivity determine the normalization.) Here the quasifree state φ is defined by the following properties:

$$\varphi(B(h_1)\cdots B(h_{2n-1})) = 0 \tag{7.2}$$

$$\varphi(B(h_1)\cdots B(h_{2n})) = \sum_P \prod_{j=1}^{n} \varphi(B(h_{P(j)})B(h_{P(j+n)})) \tag{7.3}$$

where the sum is over all permutations P of $1\cdots 2n$ satisfying $P(j) < P(j+n)$ for all $j = 1, \ldots, n$ and $P(1) < P(2)\cdots < P(n)$. For the special choice $Q = n^{-1}\sum_{j=1}^{n}\sigma_z^{(j)}$, we have the following situation. The Hamiltonian $H_0^{(N)}$ is invariant under the automorphism

$$\Theta'(a) \equiv \lim_{m\to\infty}\left(\prod_{j=1}^{m}\sigma_y^{(j)}\right)a\left(\prod_{j=1}^{m}\sigma_y^{(j)}\right) \tag{7.4}$$

which changes $\sigma_x^{(k)}$ and $\sigma_z^{(k)}$ to $-\sigma_x^{(k)}$ and $-\sigma_z^{(k)}$ while leaving $\sigma_y^{(k)}$ invariant. Hence $[\sigma_t, \Theta'] = 0$ and the unique (α_t, β)-KMS state φ must be Θ'-invariant. Since $\Theta'(\sigma_z^{(j)}) = -\sigma_z^{(j)}$, this implies

$$\lim_{t\to\infty}\lim_{N\to\infty}\overline{m}_t^{N,n} = n^{-1}\sum_{j=1}^{n}\varphi(\sigma_z^{(j)}) = 0 \tag{7.5}$$

7.2. Two-Sided Infinite Chain

Using the same notation as before,

$$\hat{\alpha}_t(a) = \lim_{N\to\infty} \exp\left(itH_0^{[-N,N]}\right) a \exp\left(-itH_0^{[-N,N]}\right) \tag{7.6}$$

is described by

$$\hat{\alpha}_t(B(h)) = B(e^{-2Jit\hat{K}_\gamma}h) \tag{7.7}$$

where $\hat{K}_\gamma$ is the same as K_γ of (6.3) except for the replacement of U and U^* by $\hat{U}$ and $\hat{U}^*$ defined by

$$(U^*f)_n = f_{n+1}, \qquad (\hat{U}^*f)_n = f_{n-1} \tag{7.8}$$

for $f = (\ldots f_{-1}, f_0, f_1, \ldots) \in l_2(\mathbb{Z})$. Here the algebra $\hat{\mathfrak{A}}$ is generated by the spins $\boldsymbol{\sigma}^{(j)}$, $j = 0, \pm 1, \pm 2, \ldots$ on the two-sided infinite one-dimensional chain, and $\hat{K}_j$ is acting on $l_2(\mathbb{Z}) \oplus l_2(\mathbb{Z})$.

Since

$$\left[H_p^{[-N,N]}, B(f \oplus g^*)\right] = 2B\left(hP^{(N)}f \oplus -hP^{(N)}g\right) \tag{7.9}$$

with $P^{(N)}f = (\ldots 0, f_{-N}, \ldots, f_N, 0 \ldots)$,

$$\hat{\alpha}_t^{(p)}(a) \equiv \lim_{N\to\infty} e^{itH^{(N,N)}} a e^{-itH^{(N,N)}} \tag{7.10}$$

can similarly be described by

$$\hat{\alpha}_t^{(p)}(B(h)) = B(e^{2it(-J\hat{K}_\gamma + hS)}h) \tag{7.11}$$

where h is the coupling constant in H_p and $S = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ on $l_2(\mathbb{Z}) \oplus l_2(\mathbb{Z})$. The unique $(\hat{\alpha}_t^{(p)}, \beta)$-KMS state $\varphi^{-\infty,\infty}$ is the quasifree state with the following two-point function:

$$\varphi^{-\infty,\infty}(B(\Gamma h_1)B(h_2)) = \left(h_1, \left(1 + e^{2\beta(J\hat{K}_\gamma - hS)}\right)^{-1} h_2\right) \tag{7.12}$$

Hence, the expectation value of

$$\hat{\alpha}_t\left(\sigma_z^{(0)}\right) = 2B(e^{-2Jit\hat{K}_\gamma}h^{(0)})^* B(e^{-2Jit\hat{K}_\gamma}h^{(0)}) - 1 \tag{7.13}$$

with $h^{(0)} = 0 \oplus f^{(0)}$ and $f_n^{(0)} = \delta_{n0}$ is given by

$$\varphi^{-\infty,\infty}\left(\hat{\alpha}_t\left(\sigma_z^{(0)}\right)\right) = \left(h^{(0)}, (2D - 1)h^{(0)}\right) \tag{7.14}$$

$$D = e^{2Jit\hat{K}_\gamma}\left(1 + e^{2\beta(J\hat{K}_\gamma - hS)}\right)^{-1} e^{-2Jit\hat{K}_\gamma} \tag{7.15}$$

(a) Case $\gamma = 0$. In this case $\hat{K}_0$ and S commute. Thus D is independent of t and

$$\lim_{t\to\infty}\lim_{N\to\infty}\overline{m}_t^{N,N} = \varphi^{-\infty,\infty}(\sigma_z^{(0)})$$
$$= \left(h^{(0)}, \left\{1 + \exp\left[2\beta(J\hat{K}_0 - hS)\right]\right\}^{-1} h^{(0)}\right)$$
$$= \left(f^{(0)}, \left\{1 + \exp\left[-2\beta(J(U + U^*) - h)\right]\right\}^{-1} f^{(0)}\right) \quad (7.16)$$

This is a strictly decreasing function of βh and hence its value for $h \neq 0$ is different from the value for $h = 0$ for $\beta \neq 0$. (h is the negative of the external magnetic field.)

By the spectral resolution of U, its explicit expression can be easily worked out:

$$\lim_{t\to\infty}\lim_{N\to\infty}\overline{m}_t^{N,N} = \frac{1}{\pi}\int_0^{2\pi}\left\{1 + e\left[-\beta(4J\cos\theta - 2h)\right]\right\}^{-1} d\theta - 1$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\tanh\left[\beta(2J\cos\theta - h)\right] d\theta \quad (7.17)$$

(b) Case $\gamma \neq 0$. By the spectral decomposition $U = \int e^{i\theta} dE(\theta)$ of the unitary operator $\hat{U}$, we have

$$\hat{K}_\gamma = 2\int_0^{2\pi}(\sigma_3\cos\theta - \gamma\sigma_2\sin\theta)\, dE(\theta) \quad (7.18)$$

with $\sigma_2 = \left(\begin{smallmatrix}0 & -i\\ i & 0\end{smallmatrix}\right)$ and $\sigma_3 = S = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ on $l_2(\mathbb{Z}) \oplus l_2(\mathbb{Z})$. Let

$$k(\theta) = \left[(2h - 4J\cos\theta)^2 + \gamma^2(4J\sin\theta)^2\right]^{1/2} \quad (7.19)$$

$$n_\theta = \left(0, k(\theta)^{-1}4J\gamma\sin\theta, k(\theta)^{-1}(2h - 4J\cos\theta)\right) \quad (7.20)$$

then $(\boldsymbol{\sigma}\cdot n_\theta)^2 = 1$ and

$$\left(1 + e^{2\beta(J\hat{K}_\gamma - hS)}\right)^{-1} = \int_0^{2\pi}\left[\left(1 + e^{-\beta k(\theta)}\right)^{-1}(1 + \boldsymbol{\sigma}\cdot n_\theta)/2\right.$$
$$\left. + \left(1 + e^{\beta k(\theta)}\right)^{-1}(1 - \boldsymbol{\sigma}\cdot n_\theta)/2\right] dE(\theta)$$
$$= \int_0^{2\pi}\left\{1 + \boldsymbol{\sigma}\cdot n_\theta\tanh\left[\beta k(\theta)/2\right]\right\} dE(\theta)/2 \quad (7.21)$$

The vector n_θ can be decomposed as $n_\theta = n'_\theta + n''_\theta$ with n'_θ proportional to $(0, -\gamma\sin\theta, \cos\theta)$ while n''_θ is orthogonal to $(0, -\gamma\sin\theta, \cos\theta)$. When (7.21) is substituted into (7.15), the term $\boldsymbol{\sigma}\cdot n'_\theta$ commutes with $e^{it\hat{K}_\gamma}$ and becomes independent of t, while the term $\boldsymbol{\sigma} \cdot n''_\theta$ acquires the time dependence $\exp[8Jit(\sigma_3\cos\theta - \gamma\sigma_2\sin\theta)]$, which vanishes in the limit $t \to \infty$ due to the Riemann–Lebesgue Lemma. Hence

$$\lim_{t\to\infty} \lim_{N\to\infty} \overline{m}_t^{N,N} = \frac{1}{2\pi} \int_0^{2\pi} m(\theta,h)\, d\theta \tag{7.22}$$

$$m(\theta,h) = -(n'_\theta)_3 \tanh\left[\beta k(\theta)/2\right] \tag{7.23}$$

The third component of the vector n'_θ is given by

$$(n'_\theta)_3 = k(\theta)^{-1}(\cos^2\theta + \gamma^2\sin^2\theta)^{-1}\left[(2h - 4J\cos\theta)\cos\theta - 4J\gamma^2\sin^2\theta\right]\cos\theta \tag{7.24}$$

Thus $\lim_{t\to\infty} \lim_{N\to\infty} \overline{m}_t^{N,N}$ is an analytic function of h and γ. Since it has a nontrivial dependence on h for $\gamma = 0$ its value for a generic h is different from its value for $h = 0$ (which is 0) for any given γ. This proves the assertion for case (B) with a general γ.

8. DISCUSSION

Although the details of our computation depend on the specific model under consideration, our method is applicable to any model, for which some type of asymptotic Abelian property can be proved. Then the return to equilibrium for a finite impurity follows from a general argument.

In the present example, the asymptotic Abelian property does not hold, but we prove a twisted version of such a property (usually found for Fermion algebras) and we presented a twisted version of the general argument.

The conclusion for impurity of an infinite size is also of a general nature for a one-dimensional system, where the surface effect is of a finite range and hence most of the impurity spins (except those near the boundary of the impurity) are in the perturbed state (at $t = 0$). Hence after the limit $N \to \infty$ with $n(N) \to \infty$ is taken, the average magnetization of the impurity is the same as the expectation value of the spin (at any point) in the uniformly perturbed equilibrium state of the *two-sided* finite chain, with the unperturbed time evolution of the same *two-sided* infinite chain. Thus the problem reduces to the case of uniform perturbation over all lattice sites, in which case the return to equilibrium is not to be expected in general.

A by-product of our investigation is about the asymptotic Abelian property. It is often used in general axiomatic arguments but it is hard to verify in individual cases. It certainly fails for "classical interaction" (i.e., the case where the interaction potential is taken from an Abelian subalgebra of α). In the present case, the asymptotic Abelian property does not hold, although a twisted version holds for $\gamma = 0$. Even the twisted version does not hold for $\gamma \neq 0$, although it holds for a (large) subalgebra.

In the present case, the twisted asymptotic Abelian property is proved on the basis of the Riemann–Lebesgue lemma. For local observables, the time dependence of the twisted commutator is of the order of $|t|^{-1/2}$ in general due to the stationary point for $\cos s$ (at $s = 0 \bmod \pi$) and hence the L_1-asymptotic Abelian property does not hold even in the twisted version. This $|t|^{-1/2}$ dependence seems to be of a general nature for a diffusion in one dimension.

Emch and Radin[14] have given a C^*-algebraic analysis of the XY model and concluded the return to equilibrium. The general background of their argument is the same as the present one. They treat the XY model on the two-sided infinite chain, in which there are no nontrivial observables invariant under the time evolution, in contrast to our case for $\gamma \neq 0$. Their conclusion is somewhat more restricted because they use the asymptotic Abelian property of the even subalgebra (the θ-invariant elements of $\mathfrak{A}$) rather than a property of the whole algebra.

NOTE ADDED IN PROOF

It has been brought to our attention that D. W. Robinson studied the twisted asymptotics of similar systems.

REFERENCES

1. E. Lieb, T. Schultz, and D. Mattis, *Ann. Phys. (N.Y.)* **16**:407–466 (1961).
2. M. Dresden, *Physica* **110A**:1–40 (1982).
3. E. Barouch, B. M. McCoy, and M. Dresden, *Phys. Rev. A* **2**:1075–1092 (1970).
4. D. E. Abraham, E. Barouch, G. Gallavotti, and A. Martin-Löf, *Stud. Appl. Math.* **50**:121–131 (1971); **51**:211–218 (1972).
5. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics II* (Springer Verlag, Berlin, 1981).
6. R. F. Streater, *Commun. Math. Phys.* **7**:93–98 (1968).
7. D. W. Robinson, *Commun. Math. Phys.* **7**:337–348 (1968).
8. H. Araki, *Prog. Theor. Phys.* **32**:844–854 (1964).
9. H. Araki, *Commun. Math. Phys.* **14**:120–157 (1969).
10. H. Araki, *Commun. Math. Phys.* **44**:1–7 (1975).
11. A. Kishimoto, *Commun. Math. Phys.* **47**:167–170 (1976).
12. H. Araki, *Publ. R.I.M.S., Kyoto Univ.* **9**:165–209 (1973).
13. H. Araki, *Publ. R.I.M.S., Kyoto Univ.* **6**:385–442 (1971).
14. G. G. Emch and C. Radin, *J. Math. Phys.* **12**:2043–2046 (1971).